NOTATIONS TOWARD A BOHM-EVERETT SYNTHESIS

MILO BECKMAN

Abstract

An explicitly constructed "multi-Bohmian" framework for the interpretation of quantum systems, based on the work of Erwin Madelung (1927).

I.
HISTORICAL CONTEXT

Here is my take on the experimental data of quantum mechanics: I think it is entirely obvious what is going on here, I think the equations tell a very clear story of the world and what it looks like and how it evolves, and I believe we have been resistant to this story in varying degrees not for any good empirical reasons, not even for any good philosophical reasons, but for the typical, mundane, human reason that we don't like what this story has to say about us, our world, and our place in the world.

I believe furthermore that the dominant dialectic in Western academic scientific-naturalist philosophy is not well equipped to handle what small-scale physics is telling us; in fact the terms of this dialectic were set with the express purpose of *avoiding* the conclusion which quantum mechanics makes inevitable, to contain the demons unleashed by Bohr, Heisenberg, Schrodinger, et al. and to do damage control as a curious public looks on; and so we are perpetually asking the wrong questions, using the wrong notations, and narrowing the scope of discourse until the obvious answer is made to look radical, wacky, and above all *unscientific*.

And yet I find that despite our best efforts to keep the undesirable truth at bay, it has nonetheless found ways to slip into the discourse, suitably subdued and dressed up in the appropriate language. I believe Bohm (1952, 1993) is exactly correct when he says that our world is made of particles which interact not only with each other but with a mysterious outside influence described by the quantum potential. I believe Everett (1956a, 1956b, 1957) and neo-Everettians such as Wallace (2012) are exactly correct when they say that the fundamental dynamics of the universe operate not on an isolated particle-world but on a larger aggregate entity. And moreover I believe that David Lewis's theory of modal realism (though obviously not tailored to the peculiar case of quantum measurement) is ultimately correct in asserting that the physical possibilities which do not obtain in our world *do* obtain, in some good, ontic sense, *elsewhere*, and that in fact their obtaining elsewhere is precisely what makes them "physically possible" in the first place.

In short I believe that Einstein, derided as he often is for his opinions on quantum theory, had all the right intuitions when he said:

> The attempt to conceive the quantum-theoretical description as the complete description of the individual systems leads to unnatural theoretical interpretations, which become immediately unnecessary if one accepts the interpretation that the description refers to ensembles of systems and not to individual systems. (1949, quoted in Ballentine 1972)

What I offer in this paper, then, is not a new interpretation of quantum mechanics; I don't think we need any more of those. What I am hoping to introduce is a set of notations and terminologies which help to clarify and collate the existing interpretations. In particular I favor the use of Madelung's ρ notation, which tells a fairly clear story of quantum reality, over Schrodinger's ψ notation, which does not. In §4 I will flesh out the ρ story by "decomposing" the wave function into a very suggestive object $\mathfrak{B} = \{w_i\}$ which looks conspicuously like an uncountable set of worlds.

I have found such clarity from using these notations that I worry they will never be adopted: they shine too harsh a light on the situation. They make manifest the core philosophical horror of fundamental physics, namely, the decentering of the human's experience of reality. We have already been forced to accept that we are animals, that we are not the center of the universe, and now we are being asked to suffer an even deeper embarrassment: that our full physical universe, everything we can plausibly have epistemic access to, is merely an insignificant and altogether unremarkable blip on the full scale of what exists. I fear that by making the choice so stark, by laying out the stakes as they actually are, I will compel some readers to simply adopt a collapse theory and get on with their purposeful lives.

The historical context here is imporant, and so here it is: Schrodinger's ψ notation is one of a few equivalent formulations of quantum mechanics introduced in the 1920s, and the fact that it managed to catch on and make its way into textbooks seems (on closer inspection) to be a matter of some historical contingency. I accept without debate that ψ is a perfectly useful tool for making empirical predictions, in other words, for doing the actual work of experimental physics and engineering. But I argue that we as philosophers have become needlessly obsessed with ψ notation, that we have essentially reduced the ontological question of the quantum state to a deep study of this object ψ, despite there being in the extant literature several other *mathematically equivalent formulations* of the fundamental dynamical law we call the Schrodinger equation.

In fact the earliest published form of the Schrodinger equation, which is to say the earliest known presentation of the predictive apparatus of modern QM, was not due to Schrodinger at all; it was Heisenberg's matrix mechanics (1925). Heisenberg had successfully resolved the relevant empirical problem—electrons were no longer falling into nuclei—but his methods were so elaborate and abstruse that contemporaries viewed them with suspicion. There was a glaring interpretative problem: Heisenberg had no answer to the basic question of *what is going on.* His approach derived predictions by applying time-*dependent* measurement operators to a time-*independent* base vector, and so there was no object in his framework which could be understood to describe at a given time the state of the system.

A year later, Schrodinger (1926a) proposed a change of basis, inverting the picture to a time-independent operator acting on a time-dependent "state" vector. Thus was born ψ, the first plausible answer to the question of quantum state. Yet Schrodinger still seemed confused and dissatisfied with his own answer, for more or less the same reason we continue to be confused and dissatisfied with "the wave function" today. This object $\psi(t)$ has a very peculiar shape, namely, in the case of a single particle, $\mathbb{R}^3 \to \mathbb{C}$. What could it possibly mean that the instantaneous state of a single-particle system is described by the assignment of a complex number to every position in space? In correspondence with Lorentz, Schrodinger suggested that "the physical meaning belongs not to the quantity itself but rather to a quadratic function of it… $\psi\bar{\psi}$, that is, the square of the absolute value of the quantity ψ" (1926b).

The following year, Madelung (1927) followed through on Schrodinger's suggestion and introduced $\rho(t)$, a new candidate for the quantum state. ρ is quite simply just $|\psi|^2$. It is a density cloud of constant total "mass" which evolves continuously and deterministically in space and time. What Madelung found (and I'll go into more detail in a few pages) is that the complex phase of ψ was not needed; this dynamical information was fully captured by the instantaneous flow velocity of the ρ cloud. In fact after this change of variables the Schrodinger equation looked so similar to the Euler equations of classical fluid dynamics that Madelung was led to describe quantum mechanics as "the hydrodynamics of continuously distributed electricity."

But his paper came too late. In the intervening months, Schrodinger's ψ had been hailed (by e.g. Planck, Lorentz, Einstein, Bohr, and Heisenberg) as a suitable resolution to the problem of quantum state; by now, Born had introduced his ontologically mysterious notion of "probability amplitude" and the statistical account; and the leading figures of the field were beginning to prosthelytize the Copenhagen school of thought in lectures and articles. By 1930 Heisenberg had published *The Physical Principles of the Quantum Theory* using Schrodinger's notation, and so Schrodinger's version of the dynamical law was canonized as "textbook QM" and the ψ notation was locked in. We can only imagine how the subsequent development of academic thought would have differed in a world where Schrodinger's original paper had used the ρ notation which he himself seems to have favored.[1]

I present this historical narrative to make the simple and I hope uncontroversial sociological point that the notations we use, in any field of inquiry, are not gospel; they are written and contested by people like you and me, and they are in the end somewhat arbitrary. And so we as philosophers, if our aim is ultimately to understand the physical nature of reality, should not concern ourselves with whatever conventions

[1]Schrodinger writes in (1926b): "What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers. ψ is surely fundamentally a real function and therefore in [the equation in which ψ is first introduced] I should be good and write a cosine instead of the exponential, and ask myself: is it possible in addition to define the imaginary part unambiguously *without reference to the whole behavior of the quantity in time*, but rather referring only to the real quantity itself and its time and space derivatives *at the point in question*." This is precisely what Madelung did in 1927.

were established in the 1920s. What these physicists discovered and first wrote down, and what we are still debating nearly a century later, still trying to understand and properly contextualize, is not any particular equation but a *statement of fact*: a dynamical law, in other words, a falsifiable claim about how the world changes in time. And that *fact* is still true, it has still not been falsifed, despite our best efforts, and so we are led to believe that that *fact* has always been true, long before we wrote it down, and of course quite independently of whether we choose to write it down in Schrodinger's notation or in Madelung's or in the supplemental notation I will in §4 introduce.

II.
AGAINST Ψ

What is the universe-object? That is the question being asked here, isn't it? Which mathematical object contains the full data of reality? That is the deeper issue at stake when we talk about notation. Analytic philosophy demands that we express all truths with mathematical precision; thus, if we aim to make universal claims about *the totality of everything*, we require a mathematical object which can be taken to represent *the totality of everything*.

This is a distinct and prior question to the actual work of fundamental ontology. If we were so lucky as to be simply handed (by some authority we trust completely) a concise data structure from which all physical facts could be read off, we would still have quite a bit of metaphysics to do. We'd have a lot to debate in terms of how to think and talk about that object, how and in what sense that object or some aspects of it could be said to exist, and how to connect the formalism back to our manifest experience of reality. But given that there are mathematical regularities inherent in the structure and dynamics of the physical universe, it seems reasonable that we'd want to know what those are, as a prerequisite for talking meaningfully about what there is.

And what excites and intrigues us about the Schrodinger equation is that it purports to offer an answer to that question: Ψ! Here is a *thing*, an algebraic object, which evolves in time, and which is sufficiently complex in its internal structure and dynamics to plausibly be *the* thing, you know, *this*.[2] (We are speaking now about Ψ, the uppercase version, an object whose shape at any given moment is Config $\to \mathbb{C}$, an altogether different object than the original ψ of Schrodinger. And we are speaking now about *the Schrodinger equation*, a description of the evolution not of a single-particle system but of any arbitrary n-particle system, including but not limited to the physical universe or any part of it, an altogether different statement than the original equation of Schrodinger.)

I am happy to accept the premise, for the full duration of this paper and all future papers I write on the philosophical foundations of physics, that the Schrodinger equation is *true*; more specifically, I accept that there is a permanent regularity or symmetry in the structure of the universe and that this regularity is well-expressed by the Schrodinger equation, interpreted in the standard way a partial differential equation is meant to interpreted; even more specifically, I accept the surrounding context that

1. The quasi-classical world of macroscopic objects like you and me can be modeled as a vast constellation of very small things called particles;
2. Any fact about the quasi-classical world can be understood as a concise summary of a (typically uncountable) set of fundamental facts about the positions, momenta, and other properties of these particles;
3. There is a continuous thing called time and at different points in time the fundamental facts differ;
4. In spite of this there are stringent correlations between the fundamental facts at one time and the fundamental facts at another;
5. And yet given any set of facts at one time, the facts at another time are *not* uniquely determined;
6. And what *is* uniquely determined is rather an assignment of a real number $p \in [0, 1]$

[2] When I say "*this*," please imagine that I am gesturing indistinctly at our general surroundings.

to a set of facts at that other time, where p corresponds (modulo some conception of what this could possibly mean) to the *probability* of those facts obtaining;

7. And the dynamical law typically written down as the Schrodinger equation provides exact guidance (modulo some handling of interaction) on how, given that set of facts at that one time, to accurately calculate p for that set of facts at that other time;

8. And moreover there is *nothing more to say* besides this; in other words, the guidance given by the Schrodinger equation represents the fullest extent to which facts at one time can be parlayed into facts at another time.

I am happy to accept all these claims, for the full duration of this paper, despite having never myself seen any of the relevant experimental evidence, for the simple reason that (more often than not, and modulo the modulos on claims 6 and 7) I personally believe these things to be true.

Given that we accept these premises, we are are suddenly rather tightly constrained on our question of what the universe-object might be: it must obey the dynamical law we call the Schrodinger equation. But we are evidently *not* so tightly constrained by these premises to be forced to accept that the universe-object is Ψ. Our choice of universe-object is in fact catatstrophically underdetermined by the empirical data.

This is not an interpretive or philosophical claim about the universe itself; it's a claim about Ψ, the mathematical object we use to represent the universe in philosophical discourse. Of course the various interpretations of quantum mechanics have substantively different takes on whether the universe is best modeled by just Ψ, or Ψ together with something else, or Ψ_A but then Ψ_B after an instantaneous discontinuity—but that's not the type of underdetermination I'm concerned with. I'm concerned with a far more basic level of underdetermination: if Ψ contains the full data of the universe, well then so does $\Psi + 1$. And so does $(|\Psi|, \arg(\Psi))$. There are an endless number of reversible transformations we could apply to Ψ, any of which would produce a new candidate for the universe-object, which is to say, a new mathematical object from which any fact of the universe can be read off. (For that matter we can also apply any number of manipulations to configuration space, the time axis, or the complex plane as well.) And we can give a variable name to any such object, and we can quite easily rewrite the Schrodinger equation in terms of that new variable, and that new form will be properly interpreted as *exactly the same statement* as the one written in Schrodinger's notation.

So to say that Schrodinger's equation is true, and to say that on some level it's *all* that is true, is emphatically *not* to say that the actual, literal equation written down by Schrodinger has any particular metaphysical significance, especially as compared with any of a number of mathematically equivalent reformulations of the same dynamical law it is understood to represent.

Granted: it shouldn't matter (for the purpose of metaphysics) which version of the object we choose to focus on, which object to name with a single letter; if they are all interchangeable, then all the same metaphysical claims can be identically expressed in terms of whichever. Sure, we could let $\Phi = -\Psi$ and we could rewrite the dynamical law in terms of Φ and we could philosophize about the ontological nature of Φ, but at the end of the day that isn't much different than philosophizing about the ontological nature of Ψ, and ultimately to say that Φ is a real, fundamental entity is probably no different on the final analysis than to say that Ψ is a real, fundamental entity.

So it *shouldn't* matter, no, but I think it's clear to see that in practice it does. I think it's undeniable that the notations we use at the very least influence and constrain the ontologies we come up with. So much ink has been spilled over the notion of assigning at each time a complex number to every conceivable configuration of reality, in other words, over the shape

$$\Psi : T \times \text{Config} \to \mathbb{C}$$

when (we will soon see) under Madelung's paradigm we would have instead been spilling ink over the notion of assigning at each time a *density cloud* to every conceivable configuration of reality:

$$\rho : T \times \text{Config} \to \mathbb{R}^{\geq 0}$$

and (we will soon thereafter see) under my proposed alternate framing we would have instead been spilling ink over the notion of assigning to each time and *index* a unique configuration of reality:

$$\mathfrak{B} : T \times I \to \text{Config}$$

The point I am trying to make is not that there is anything inherently or intrinsically at stake, ontologically speaking, in our choice of answer to the underdetermined question "what is the universe-object?" The point is that we are perpetually suffering from a failure of imagination, ontologically speaking, due to the manifest inscrutability of the answer we have by convention adopted.

Also granted: the Schrodinger equation as written by Schrodinger is quite elegant and parsimonious, which might recommend Ψ over a candidate object whose corresponding dynamical law is more cumbersome. We should certainly privilege Ψ over something absurd like $\Psi + 1$, since the dynamical law written in terms of this latter variable would include two extraneous instances of "-1" and would clearly indicate a poor choice of variable. It doesn't seem obviously wrong to me that a notationally concise expression of a fact might be in some good sense more fundamental or more informative about the nature of physical reality than a convoluted expression of that same fact.

While this argument supports Ψ over the vast majority of variables we could equivalently choose, it does not in my opinion support Ψ over Madelung's ρ. If anything, this is exactly the argument in favor of ρ. For one thing, there is the obvious point that $\int \rho$ is a more direct and elegant expression than $\int |\Psi|^2$, especially if we are to understand this value as representing some kind of probability, and *especially* given that this expression is *the only bridge* connecting the formalism back to experimental data, in other words, given that this expression is the only thing we genuinely know to have some manner of physical basis in reality.

And if we must compare dynamical laws, well, I think it's fair to say that Madelung's equations (again, a more detailed treatment is coming) are not too bad:

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\rho \nabla \cdot \vec{u} \\ \frac{D\vec{u}}{Dt} &= -\frac{1}{m}\nabla(Q+V)\end{aligned}$$

This expression of the fundamental dynamical law bears an exact resemblance to the Euler equations of classical fluid dynamics, giving us a salient and intuitive picture of what ρ is, namely, an incompressible fluid of constant total "mass" shifting in space under the influence of classical potential V and quantum potential Q.

After all, Madelung's ρ is not (like $\Psi + 1$) some nonsensical choice of variable with no theoretical motivation. It was developed specifically with the relevant physical circumstances in mind. And come to think of it, as familiar as it is to us today, Schrodinger's form of the dynamical law is ultimately pretty *weird*, a famously opaque and esoteric thing, and if you really try to imagine that you had grown up with ρ and you had always thought of quantum physics in terms of ρ, writing everything in terms of $\sqrt{\rho}\, e^{i\theta}$ begins to look like some nonsensical choice of variable with no theoretical motivation.

And finally, *granted:* the physicists use Ψ, the experimentalists use Ψ, Ψ seems to be the efficacious choice for doing the actual work of fundamental research and quantum engineering. Yes, well, I've spoken to the experimentalists, and they have assured me repeatedly that they are using Ψ for its original intended purpose, which is to say as a "means for predicting probability of measurement results... somewhat as laid down in a catalog." They take the predictive apparatus of quantum mechanics to be exactly that, a predictive apparatus, a black box of sorts, a thing you do on paper to obtain the necessary numerical values. They compute with Ψ for the same reason they employ a naïve and underspecified Copenhagen framework: because (1) it works, and (2) it's what everyone else is doing. For their purposes, this is all that is needed. Said Bohr in defense of Ψ and the Copenhagen approach, “It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature” (quoted in Aage 1963).

It’s worth considering that philosophers might benefit from a different mathematical framework better suited to *our* purposes, namely, understanding what the world is. I have no doubt that textbook QM is perfectly fine for the LHC and for WMDs and I do not intend to storm into CERN and insist they use ρ and $\mathfrak{B}$. The experimentalists are doing just fine. It’s the philosophers of physics who seem perpetually confused, and it’s the philosophers of physics who I believe could stand to benefit from turning the object over in our hands and looking at it from a new angle.

III.
MADELUNG’S NOTATION

Let’s return for a moment to a single-particle case. Consider the case which launched the development of modern quantum theory: a single electron in a Coulomb potential, in other words, a hydrogen atom.

If (as was once believed) an electron is a classical point-object which obeys Newtonian laws of motion, then its trajectory $x(t)$ through potential $V(x)$ is determined by

$$\frac{d\vec{v}}{dt} = -\frac{\nabla V}{m}$$

An initially stationary classical point-electron will oscillate around the nucleus, radiate energy, and eventually fall into the nucleus.

Now consider another incorrect model: imagine an electron is a classical *fluid*. Yes, imagine that we are dealing with some small, negatively charged, compressible cloud of electric fluid. In classical fluid dynamics, we’d model this as a continuous density cloud $\rho(x,t)$ with constant total $\int \rho$. The evolution of this density cloud in time is determined by the initial $\rho(x)$ and flow velocity $\vec{u}(x)$ together with Euler's conservation equations:

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \vec{u}$$
$$\frac{D\vec{u}}{Dt} = -\frac{\nabla p}{\rho} - \frac{\nabla V}{m}$$

The top equation is a continuity equation, expressing conservation of mass; the bottom is conservation of momentum at each point,[3] under the influence of some internal pressure gradient $p(\rho)$ and an exogenous (in this case electric) potential $V(x)$. Compare this balance of momentum equation to the Newtonian force law for the point-electron case.

An initially stationary and reasonably distributed classical electron-fluid will fall towards the nucleus, slosh back and forth around it, radiating energy (it's electrically charged, after all). What happens next depends on the internal mechanical pressure p. If the electron-fluid is infinitely compressible, that is if $p(\rho) \equiv 0$, the fluid will all eventually collect into the nucleus-point. If p is instead some reasonable monotonic function of ρ, the fluid will in the limit settle into a spherical density gradient around the nucleus, forming a sort of microscopic gas planet.

Enough imagining. Now consider the accepted model: an electron as a quantum-mechanical object, which is to say, an object which obeys the Schrodinger equation, which is equivalently to say, an object which obeys the Madelung equations. Then what Madelung would call the "quantum state" is given by a continuous probability density cloud $\rho(x,t)$ with instantaneous flow velocity $\vec{u}(x,t)$, and the time-evolution of the quantum state is given by

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \vec{u}$$
$$\frac{D\vec{u}}{Dt} = -\frac{1}{m}\nabla(Q + V)$$

This system of equations is—I really can't stress this enough—mathematically equivalent

[3] Note that this equation uses the material derivative, $\frac{Dy}{Dt} := \frac{\partial y}{\partial t} + \vec{u} \cdot \nabla y$. This indicates the rate of change for a point *following the flowing substance*, not at a point fixed in space.

to the time-dependent Schrodinger equation for a single non-relativistic particle written in position basis. It is derived by the polar substitution $\psi = \sqrt{\rho}\, e^{i\theta}$ followed by the change of variable $\vec{u} = -\frac{\nabla\theta}{\hbar m}$. A step-by-step derivation is given in Appendix A.

This is, to me at least, an eminently sensible way to write the equation of motion for a quantum-mechanical object. By sensible I mean it *makes sense*: it offers a clear and vivid picture of what kind of thing is going on here. You don't need to believe that an electron is literally a shimmering yellow fluid to see that this is at least a useful conceptual analogy: *electron as density cloud.* The dynamics are easy to imagine and predict, because they are so similar to the dynamics of another well-known system: a three-dimensional compressible gas cloud. If you can picture how a 3D fluid would fall through a potential gradient, you can picture more or less how an electron's "wave function" evolves in time.

In fact, while it's common post-Bohm to write the second Madelung equation as above in terms of Q, the quantum potential

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}$$

we could instead introduce a new variable

$$p_Q = -\frac{\hbar^2}{2m^2}\sqrt{\rho}\,\nabla^2\sqrt{\rho}$$

and rewrite the second Madelung equation as

$$\frac{D\vec{u}}{Dt} = -\frac{\nabla p_Q}{\rho} - \frac{\nabla V}{m}$$

which is *exactly* the classical Euler equation for a fluid with mechanical pressure p_Q. Admittedly p_Q looks like an awfully weird equation of state,[4] but there you have it.

I have to stress again that I am not (yet) making any interpretive or philosophical

[4]This is an understatement. The value p_Q is a function not only of the local density ρ but its instantaneous local curvature. This is unlike any classical fluid or substance I am aware of.

claims. It may be tempting to read this discussion as an endorsement of a particular view about the nature of electricity and quantum objects. It is not. So far I am merely shifting the language of discourse, from the dominant academic convention to an alternate and mathematically equivalent framework, that of Madelung (1927), suitably updated to imitate the contemporary conventions of classical fluid dynamics. This will motivate the eventual interpretive and philosophical moves I make in §5–7; until then we are simply playing mathematical games with notation and terminology. If you follow the manipulations which lead to Madelung's equations and find the result interesting or worthy of further exploration, then you agree with me so far.

With that proviso in mind, here are the terminologies. This function $\rho(x,t)$ has the form of a standard physical density function, so we'll call it the *density function* or *density cloud* for the position of the particle at time t. It's tempting to use a term like "probability distribution" or "probability cloud" because (1) the condition $\int \rho \, \mathrm{d}x = 1$ is reminiscent of things we call "probability distributions" in the statistical sciences; (2) the value of $\int_R \rho \, \mathrm{d}x$ over a region of space corresponds experimentally to the observed statistical frequency of finding the particle in that region;[5] and (3) the common practice in QM is to refer to this quantity in probabilistic or pseudo-probabilistic terms. But I do not wish to take sides on the nature of this ρ or its ontological basis or its connection with the standard philosophical concepts of chance and credence, so I will stick with the more agnostic term *density*.

And we will refer to the vector field $\vec{u}(x,t)$ as the *flow velocity* of ρ, in accordance with the terminology of classical fluid mechanics. $\vec{u}$ is not really a new object so much as a description of how the density cloud $\rho(x)$ changes in time. In fact if we have complete knowledge of ρ on any continuous time interval, even just a picosecond, we can recover $\vec{u}$ on this interval from the first Madelung equation, and then propagate out the behavior of ρ for as long as the dynamical law continuously applies. Thus we hold onto $\vec{u}$ for the

[5] Throughout this discussion we are ignoring the phenomenology of interaction and collapse, which will be discussed briefly in §6. Assume unless otherwise specified that we are talking about a time interval on which the dynamical law continuously applies.

same reason we do in Eulerian fluid dynamics, and for the same reason we hold onto $\vec{v}$ in Newtonian mechanics: to save the *dynamical* state of the system at a fixed time t_0, so that the future or past can be projected from t_0 alone.

This raises the (still terminological) question of whether to refer to ρ alone or $(\rho, \vec{u})$ together as the *quantum state* of the system at a given time t_0. In deference to contemporary convention I will refer to $(\rho, \vec{u})$ as the quantum state, so that ψ can be uniquely recovered from an instantaneous snapshot of $(\rho, \vec{u})$; in other words, so that the Schrodingerian "quantum state" is in one-to-one correspondence with the Madelungian "quantum state." But it should be noted, for when we do finally get to interpretation, that there is no need for $\vec{u}$ to have any independent basis in reality aside from as a descriptor of the instantaneous rate of change of ρ.

This point is important and bears repeating in Schrodinger's language, for those of us who grew up on ψ. What I am saying is that over any continuous interval of time, *the complex phase of ψ is not a distinct piece of information from the magnitude of ψ.* Complete knowledge of $|\psi|$ over any continuous interval of time on which the Schrodinger equation applies grants complete knowledge of ψ over that same interval. It's only at a fixed instant in time t_0 that knowledge of $|\psi|$ is insufficient to project the future and past; the complex phase at t_0 stores *dynamical* information about the system. It's essentially a mathematical bookkeeping trick that makes the linear algebra work out nicely.

This observation I take to be Madelung's primary contribution to the dialectic. What Madelung noticed is, to put it bluntly: θ doesn't matter. Over any continuous interval of time on which the Schrodinger equation applies, $|\psi|^2$ is a *complete dynamical description of the system.* Any fact which can be read off of $\psi : T \times \mathbb{R}^3 \to \mathbb{C}$ can be read off of $\rho : T \times \mathbb{R}^3 \to \mathbb{R}^{\geq 0}$. This is an agnostic mathematical point, to which all parties to the interpretation discussion will have to agree. And this is the point that Madelung's ρ notation makes manifest: the quantum state at t_0 is given by $(\rho, \vec{u})$, but the evolution of the system over time is fully described by a shifting density cloud, ρ.

There's one last feature of Madelung's framework that I need to point out before

I introduce my own supplemental notation. It may seem unmotivated, but it's a very important point; it will actually turn out to be the centerpiece of my notation.

Consider one last time the case of a single electron in a Coulomb potential. We understand the system to be well-modeled by an Eulerian fluid with a peculiar equation of state. Now we ask the question: as this density cloud shifts, what is the path of a hypothetical particle suspended in it? Ignore, for a moment, the evident absurdity of the question. This is a pure intellectual curiosity. Place a point at x_0 at time t_0 and allow it to flow with the fluid, $\vec{v}(t) = \vec{u}(x(t), t)$. What trajectory will it trace out?

The answer, as many have pointed out, is the unique Bohmian trajectory from (x_0, t_0).

$$\frac{d\vec{v}}{dt} = \frac{D\vec{u}}{Dt} = -\frac{1}{m}\nabla(Q + V)$$

This should not be a surprising result. Bohm's guidance equation is constructed explicitly so that a statistical ensemble of particles scattered according to $\rho(t_0)$ will continue to replicate the evolution of ρ as time proceeds. They are made to dance in this peculiar way, on Bohm's telling, by the quantum potential term Q. But whether we write this into the equation as a potential Q, or as a quantum force F_Q, or as a quantum pressure p_Q, the resulting trajectories are of course exactly the same.

Madelung and Bohm are in the end making the same point with different language: at small scales, we require a very peculiar correction term to the Newtonian equations of motion, and this correction term is a function of ρ and its local curvature. They are just coming at the story from opposite directions. To Madelung, a "Bohmian trajectory" is merely the path of an infinitesimal parcel of the density cloud; to Bohm, the "density cloud" is nothing more than a continuous ensemble of hypothetical Bohmian particle-trajectories.

I find the analogy to classical fluid dynamics illuminating, so here it is again. A classical fluid can be thought of either as a continuous density cloud or as an ensemble of discrete point-particles with definite locations. Madelung is focused on the continuous density cloud, and Bohm is looking at the point-particles, but they are ultimately looking at the same object. This pair of perspectives is the motivation for my notation.

IV.
MY NOTATION

Back to the n-particle case: our physical universe. We will continue to assume we are working on an interval of time T on which the dynamical law continuously applies. And we will assume that a quasi-classical worldstate is given by a point in configuration space, $c \in \text{Config}$, whatever that space may be. And so, using Madelung's notation (which is the basis for my supplemental notation), the quantum evolution of the universe over this time interval is completely described by a map

$$\rho : T \times \text{Config} \to \mathbb{R}^{\geq 0}$$

which obeys the Madelung equations on T.[6]

There is something very evocative about this map. It has the appearance of a *density cloud shifting in configuration space.* In the classical picture of reality, each time t is mapped to a configuration $c \in \text{Config}$, a single complete state of the world. In this quantum picture, each time t is instead mapped to a continuous distribution of configurations, with greater density near some than others. The analogy to classical fluid dynamics raises a natural next question: can we decompose this "Madelung fluid of the universe" into individual point-worlds?

Yes. In fact, when we decompose the Madelung fluid into infinitesimal parcels and press play, each parcel traces out a consistent Bohmian history of the world. This is, again, not magic: Bohm's equation of motion (3.8) in his (1993) is exactly the Madelung balance of momentum equation restricted to a fixed initial x_0.

[6] I have made the difficult decision here to still use a lowercase ρ in the universe case, despite distinguishing Ψ from ψ and $\mathfrak{B}$ from B. This is because uppercase P is the exact same glyph as the latin P, which suggests a probability rather than a density. Because ρ is often interpreted as a probability density, the potential for confusion is too great.

This is the point my notation is intended to draw out. I will now rewrite the universe-object as *a continuous ensemble of Bohmian trajectories*, $\mathfrak{B} = \{w_i : T \to \text{Config}\}$.

A very similar move is made by Charles Sebens (2015), who also seeks to analyze "the wave function" as a collection of Bohmian worlds. His approach is to scatter a very large finite number of points at random according to ρ (he calls it $|\Psi|^2$), so that the *fraction* of all points in a region $R \subseteq \text{Config}$ roughly tracks $\int_R \rho$. This approach takes the fluid analogy to the logical extreme: a classical fluid is (after all) composed of a large finite number of particles. So Sebens argues for an ontology in which these discrete Bohmian point-worlds are fundamental, and ρ or $|\Psi|^2$ is (as in classical fluid dynamics) merely a continous approximation.

My approach differs in one crucial respect. I instead decompose ρ into an uncountable set of points with a measure, so that the *measure* of the set of points in a region $R \subseteq \text{Config}$ will exactly equal $\int_R \rho$. I choose this approach because I want my resulting universe-object to be *exactly equivalent* to the standard Ψ. I want, in other words, to demonstrate constructively that "the wave function" is informationally equivalent to a continuous ensemble of Bohmian trajectories.

We define the desired object:

Definition. A *bundle*[7] is a set $B = \{w_i : T \to \text{Config}\}_{i \in I}$ of continuous paths through configuration space, equipped with a measure $m : \Sigma_I \to [0, 1]$.

We can think of m as the "mass" of a subset of B. We will often want to know what "mass" is in a region R at time t, so for a given B we introduce the shorthand

$$m(R, t) := m(\{i \in I \mid w_i(t) \in R\})$$

This quantity is to correspond with $\int_R \rho = \int_R |\Psi|^2$, what on the statistical account corresponds to the "probability" of measuring the system in a configration $\in R$ at t.

[7]Not to be confused with the *fiber bundle* of topology.

Definition. Bundle B *models* continuous density function $\rho : T \times \text{Config} \to \mathbb{R}^{\geq 0}$ if

$$m(R, t) = \int_R \rho(c, t) \, \mathrm{d}V$$

for all Borel sets $R \subseteq \text{Config}$ and $t \in T$. We denote this relation $B \vdash \rho$.

Claim (Existence of B given ρ). If a function $\rho : T \times \text{Config} \to \mathbb{R}^{\geq 0}$ is continuous in T and Config, and satisfies $\int_{\text{Config}} \rho(c, t) \, \mathrm{d}V = 1$ at all times $t \in T$, there exists a bundle B such that $B \vdash \rho$.

Claim (Uniqueness of ρ given B.) If B is a bundle and $B \vdash \rho_1$ and $B \vdash \rho_2$, then $\rho_1(c, t) = \rho_2(c, t)$ at all c, t.

An informal constructive proof of these claims is outlined in Appendix B, for the relevant special case where $\text{Config} = \mathbb{R}^n$. The strategy is to borrow the tools of fluid mechanics and define the local flow $\Phi : \text{Config} \times T \to \text{Config}$ from the velocity field $\vec{u}$. I hope the intuition behind these claims is clear without formal proof.

Taken together, these claims allow us to switch back and forth losslessly between two pictures: that of a density cloud shifting continuously in a space, and that of a measure space of individual paths through that space.

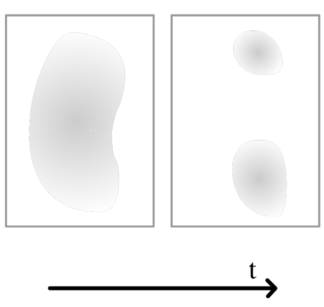

Figure 1: ρ picture

Figure 2: B picture

That is basically it. Let's take stock.

We have just argued for the existence of an object $\mathfrak{B} = \{w_i\}$, a set of time-indexed paths through configuration space, which contains the full data of the quantum evolution of the universe on T. $\mathfrak{B}$ is nothing more than an "inverted" way of looking at ρ or $|\Psi|^2$, by tracing the paths of hypothetical points suspended in the imaginary fluid. ρ is uniquely recoverable from $\mathfrak{B}$, and Ψ is uniquely recoverable from ρ, and so if Ψ is a complete description of reality, well then so is $\mathfrak{B}$.

And really, take a look at this object!

$$\mathfrak{B} : T \times I \to \text{Config}$$

This seems to me an eminently sensible way to look at the data of QM, by which I again mean that it *makes sense*. Our observed experience of reality, the so-called manifest image, simply falls out of the formalism: it looks to be one of the w_i's. Each index $i \in I$ is associated with a coherent history of the world w_i, which we may refer to as a *Bohmian trajectory* or a *world-path*, or (more to the point) a *world*, or (to borrow the popular term) a *timeline*.

I am, to re-reiterate, *not yet making any interpretive or philosophical claims*. I am not arguing, as Sebens does, that this set of Bohmian world-paths is what fundamentally exists and that Ψ is an illusion. I am just turning the universe-object over and presenting it from a new angle. Anything you could say about Ψ, any metaphysical stance you might wish to take, you could just as well express in the language of ρ and $\mathfrak{B}$, and vice versa. Of course, some claims may be easier to express in Ψ notation than in ρ or $\mathfrak{B}$ notation, and vice versa.

Which brings me to the very point I came here to make:

The metaphysical claims of many-worlds and pilot-wave theorists are far more easily and clearly expressed in terms of ρ and $\mathfrak{B}$ than they are in terms of Ψ.

To put it sharply:

If a many-worlds or pilot-wave theory turns out to be correct, then Schrodinger's Ψ *notation is a confused way of talking which serves to obfuscate the true nature of reality.*

And to put it colorfully:

We might be catastrophically misled as to the nature of reality, as an academic community, if we unfortunately found ourselves on a dystopian alternate trajectory w_i *where* Ψ *notation became the only acceptable way to talk about the universe-object.*

This thought should motivate, if not a wholesale rethinking of the constrictive approach we take to the project of "wave function ontology," at least a subtantial opening of new doors. We of course don't need to discard the existing discourse, but we should certainly make room for conversations about quantum reality which take alternate perspectives on the object of interest.

The next three sections discuss the interpretations of quantum mechanics. As signaled throughout, I will end up favoring a sort of streamlined Bohm–Everett joint approach, but (whatever your religion) I hope you will find something illuminating or clarifying about the reasoning that brings me to that conclusion.

V.
CONFIGURATION SPACE IS NOT FUNDAMENTAL

The first thing to say about interpretation is something I'm amazed I have to say at all: configuration space is not a real, physical space. Because all of these universe-objects we're talking about sit in configuration space, it's tempting to call it a day and say that configuration space is *the* space, and one of these mathematical objects is a *physical* object, and that's all there is to it. But this is tempting like an afternoon nap. It doesn't mean it's a responsible way to approach the task at hand.

I'll say it: I don't think anyone really believes configuration space is fundamental. I think the people who argue for configuration space as a real, physical space don't actually believe what they are arguing and simply haven't found anything better to argue yet. We write down these first-pass candidate ontologies because they are straightforwardly suggested by and self-evidently consistent with the underlying formalism. Yes, it is undeniable that these are empirically adequate candidate theories. Sadly, they are also *absurd* candidate theories, depicting a reality which is conspicuously not this one.

The patent absurdity of a candidate space ontology is evinced by its basic failure to answer a question like, *Where is my shoe?* So you can procure accurate predictions for the outcomes of experiments, wonderful, but *where is my shoe?* Don't tell me to read it off the state vector, I said *where* is it? If your principled stance is that Config is the only space which is physically instantiated, then please find me a location in that $3n$-dimensional space where my actual, physical, three-dimensional shoe, the one I am currently looking at, about 33cm $\times$ 15cm $\times$ 11cm, is sitting. I am made to understand that this is not a popular view among philosophers of physics, but yes: I demand that the universe must contain, as a physical subspace, *this*.

I am perfectly happy to accept that things are not as they seem. The Newtonian model of the world makes the rather shocking assertion that what appears to be my shoe is in fact a vast constellation of some zillions of tiny vibrating dots. But these dots are still located in $\mathbb{R}^3$ and they still (taken as a collective) have the shape of my shoe. It's a counterintuitive model of the world, no doubt, but it *is* a model of the world. It's a precise mathematical description of a 3D world which, if arranged and set in motion exactly as described, would look like *this*.

A configuration space ontology is a different sort of thing entirely. This view posits not that my shoe is *composed of parts* but that it is *essentially illusory*, that there simply *does not exist a structrure*, in the final analysis of what is and what isn't, which has the physical shape of my shoe. This is not, as sometimes suggested, a mere difference in degree, a new stage of counterintuitiveness we must with epistemic humility accept. This is an obstinate refusal to answer the question that was asked. I take the purpose of philosophical discourse to be, on some level, to figure out *what is going on here*, to reach an understanding as to the nature of all *this*. So to say that *this* is an illusion, or that *this* is "virtual" or "mere commentary" or what have you, is to miss the point entirely. We are asking for someone to explain *this*, to provide an consistent and coherent mathematical description of *this physical reality*,[8] and "a dot and/or cloud moving in configuration space" is a manifestly bad answer to that question, one that leads the asker suspect the answerer either doesn't know or would prefer to ignore the actual answer.

I mean, really, configuration space has (on the standard telling) three dimensions *for every particle*. What in the world are you *talking* about? So you mean to tell me that my shoe is in actual fact a three-zillion-dimensional Euclidean space inhabited by a shivering infinitesimal point and/or wafting cloud of ethereal goop? I'm afraid I don't think that's true. I'm afraid what you're referring to is, well, a *configuration space*, in the original sense, meaning, a topological space which describes the possible configurations of an object or system. The configuration space of a six-sided die is $\{1, 2, 3, 4, 5, 6\}$. The configuration space of a Ferris wheel or hand-crank is S_1, the circle. And the configuration space of the universe is $\mathbb{R}^{3n}$. Fantastic. That still leaves the question of what the universe actually is.

Whatever is going on in configuration space has the character of a *data table*, of a manner of tabulating and tracking the positions of the various components of a system, "somewhat as laid down in a catalog." It is distinct from the actual system, as a sketch of a bird is distinct from an actual bird. The vision of a cloud in configuration space depicts quantum reality in exactly the same sort of way the vision of a point in phase space depicts a (hypothetical) Newtonian reality. But no one would say that a Newtonian reality *fundamentally is* a point in phase space, because that's ridiculous.

So I roundly reject two interpretations of the preceding formalisms. I reject the pseudo-Bohmian "marvelous point" view that one w_i exists, physically as a point in $\mathbb{R}^{3n}$, and it is jostled around in that space by a fluid which also exists. And I reject the pseudo-Everettian view that *only* that fluid exists, or that every w_i exists and they

[8] Now please imagine that I am also rapping my knuckles emphatically on a table or wall.

are fundamentally indistinguished from each other, comprising a single unified physical entity that takes the form of a high-dimensional fluid. And I label both of these views "pseudo" because, if you read the actual words of Bohm and Everett, this is quite clearly not what either man had in mind.

The only reason these configuration space ontologies retain currency is because, unlike in the Newtonian case, there's not an obvious story to tell about the actual system our configuration space is describing. In the Newtonian case, "a path through phase space" is quite legible: we're tracking the changing state of the world. In the quantum case, we instead have a *density cloud* or *distribution* of constant total $\int \rho$ shifting in configuration space, so it's not so obvious what we're tracking here.

Or is it? I know we're supposed to act confused, but isn't it fairly clear what we're modeling here? It looks like we're keeping tabs on how *much* of something, what *measure* or *mass* or *proportion*, is in what state at what time. It looks like *something-or-other* is continuously distributed across possible configurations of reality, and that the distribution changes continuously in time. Is that not the obvious take on this data?

That's certainly what it looked like to Schrodinger when he first presented ψ in his original 1926 paper:

> The [one-electron] wave function physically means and determines a continuous distribution of electricity in space. ... Now how are these conceptions to be generalized to the case of more than one, say of N, electrons? ... The real continuous partition of the charge[9] is a sort of mean of the continuous multitude of all possible configurations of the corresponding point-charge model, the mean being taken with the quantity $\psi\bar{\psi}$ as a sort of weight-function in the configuration space.

He declines to specify whether the "continuous multitude" is ontic or merely epistemic, but he certainly sees the formalism as representing a weighted average of quasi-classical world-states. (In other words, a set $\{w_i\}$ with measure.)

[9] He refers to ρ as a "charge" in analogy to the one-electron case.

And here is Born, quoted in (Ballentine 1972):

> To say that ψ describes the "state" of one single system is just a figure of speech, just as one might say in every day life: "My life expectancy (at 67) is 4.3 years!" ... what it really means, of course, that you take all individuals of 67 and count the percentage of those who live for a certain length of time. This has always been my own concept of how to interpret $|\psi|^2$.

And here is Einstein at the 1927 Solvay Conference, making the point explicit:

> The deBroglie-Schrodinger waves do not correspond to a single electron, but to an electron cloud, extended in space. The theory does not give any information about the individual processes, but only about an ensemble of an infinity of elementary processes.

To which Bohr, representing the Copenhagen contingent, could only respond:

> I don't understand what precisely is the point which Einstein wants to [make] ...[the] theory is nothing else [but] a tool for meeting our requirements and I think it does. ... I think we are dealing with some mathematical models which are adequate for description of our experiments.

I quote these men not because I think we ought to take their words at face value, but to affirm that what I'm seeing is really there, that the obvious interpretation of ρ or $|\psi|^2$ is as a continuous distribution of many quasi-classical three-dimensional system-states, whether ontic or merely epistemic; and moreover that the alternative picture of ψ as somehow representing a single inscrutable state of affairs was developed and intended as a "figure of speech," as a "tool for meeting our requirements," and that the laborious attempts to ontologize ψ began only after a few generations of physicists had been instructed to understand individual quantum systems in terms of ψ.

There is a genuinely substantive, non-sociological reason why the original approach of Schrodinger, de Broglie, and Einstein struggled to win adherents: the failure of the *epistemic* ensemble view. This is the view for which Einstein is justifiably criticized. If ρ

represents an *epistemic* distribution over world-states, only one of which actually obtains in nature, then a deeper theory should be able to identify which individual system is the "correct" world and moreover should be able to predict its future evolution without reference to the "incorrect" worlds. But this is of course the view of "local hidden variables" which was elegantly disproven by Bell in the 60s.

Put another way, if we try to understand the electron clouds of Schrodinger as *epistemic* ensembles, describing our state of knowledge as to the actual point-position of an actual point-electron, we run into the problem that these clouds physically self-interact, in a way that epistemic ensembles of purely hypothetical particles really should not.

And put another way, when Bohm tried to formalize the suggestions of de Broglie and Einstein a few decades later, isolating one w_i as real and characterizing ρ as merely epistemic, he found that the deterministic time-evolution of w_i was irreducibly dependent on ρ, curiously enough, by way of the quantum potential term Q.

And all these failures of the *epistemic* approach seem to support instead a straightforward *ontic* or *realist* approach to quantum universe-object:

1. ρ describes a continuous aggregate of quasi-classical worlds that look like *this*.
2. The deterministic laws of physics operate at the level of that aggregate.
3. Therefore we as physicists have to keep tabs on *all* the worlds and not just *this* one (we have to maintain a density map on configuration space) because the evolution of *this* world depends on the "density" of worlds like this one, by way of that pesky quantum correction term Q.

Maybe that's simply too rich for your blood. Before you go grasping for an unsavory collapse postulate, let me offer you an easier out.

VI.
BOHM AND EVERETT MOSTLY AGREE

We might imagine Bohm coming onto the scene in 1952, in this alternate timeline where Madelung's notation is the canonical one, presenting his same ontology in different terms. "The equations of Madelung," writes this fictional Bohm,[10] "are understood to determine the motions in configuration space of a hypothetical fluid ρ, representing the evolution of an isolated quantum system, such that the 'mass' of fluid in a region of configuration space corresponds exactly to the observed frequency of measuring that system in such a configuration. In this article we shall present an ontological interpretation of the quantum theory, offering a precise account of the behavior of an individual system at the quantum level, and explaining the emergence of Newton's laws of motion in the classical limit."

This otherworldly Bohm would have far less work to do than our own Bohm. Given the decomposition of ρ into flow-paths $\{w_i\}$, he would simply have to note that along each w_i the motions of individual particles are given by

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\frac{1}{m}\nabla(Q + V)$$

and demonstrate that the potential Q is vanishing in the limit where $\hbar \to 0$, yielding Newton's law. He would then go on to assert that only one such w_i actually exists (this one) but "we have, in practice, a statistical ensemble with probability density $\rho(c)$. The use of statistics is, however, not inherent in the conceptual structure, but merely a consequence of our ignorance of the precise initial conditions."

In short, the theory of Bohm can be summarized as:

One w_i actually exists and the rest are imaginary.

This is, to my mind, a very good answer to the quantum predicament. It is undeniably curious that the time-evolution of each w_i is dependent on ρ, but the point is not fatal;

[10] Portions of these quotes are adapted from his actual (1952).

we introduce the pilot wave and the guidance condition and move on with our days. For the relevant task of *locating the individual quantum system which actually obtains* and confirming its continuity with classical mechanics, Bohm's contribution is invaluable.

In fact I would have to say that, for all practical purposes, I am myself a Bohmian. When I engage in any philosophizing which does not directly pertain to the details of physical law, I do generally consider myself to live in an isolated three-dimensional universe wherein all objects have definite positions and properties. This is, as they say, the obvious ontology evolving in the obvious way. Whether the other w_i's do in some broader sense "exist" is an interesting question, worthy of some introspection, but it's a question which in nearly every context can be factored out of the equation by focusing our attention on a single w_i—this one.

For instance, my ethical considerations are certainly limited to the beings of *this* universe, whether or not this universe should turn out on the final analysis to be one of continuously many. I will not adjust my behavior to attempt to send ripples through the ρ-cloud and improve the quality of life of beings on alternate world-paths. Perhaps this is on some level a parochial attitude to take, but alas, what ever *I* am, *I* appear to be confined to this one world-path w_i, and my thoughts and cares appear to be largely confined to this one world-path w_i. If it's any consolation, I do try to make my ethical decisions with the beings of *all world-paths consistent with my knowledge* in mind; if I am unsure whether an atom will decay, I will not put the cat in the box.

Moreover, in everyday usage and in most cases of formal usage, I apply the "exists" predicate only to objects of this w_i. I would not consider dragons or unicorns to exist, even if there were shown to be an alternate evolutionary timeline consistent with the laws of physics in which creatures well-described by these terms came to populate the Earth. And I would consider the term "Eugene McCarthy's vice president" to be one without a referent in nature, even if it does resolve to a specific individual on some now-distant patches of the ρ-cloud.

And I suppose if you forced my hand, if you asked me to explain *how* the guidance condition is physically realized, *why* all the point-particles of our world appear to be nudged from their classical trajectories in a globally correlated way, I would have to say the best answer is probably that *this* is one slice of a larger picture.[11] I would have to say that "the pilot wave" (or whatever it is that ρ describes) is probably a physical entity of some kind, which bears the relation to our w_i that an ocean wave bears to one of its constituent molecules. And I suppose I would have to say I find it hard to imagine that we are uniquely privileged among the $\mathbb{R}^3$-slices of this object with the gift of concrete instantiation, somewhat like Leibniz's vision of the optimal world uniquely actualized in the mind of God. Our w_i seems far too arbitrary and bad for such a thesis to be plausible.

But ultimately this question is a facsimile of the familiar Lewis–Stalnaker debate on modal realism, if we take the other w_i's to be "possible worlds" in the standard modal sense. The debate centers on whether the word "actual" (in the context of a phrase such as "the actual outcome") is an indexical term like "here" or a non-indexical term like "Cincinatti" (see e.g. Stalnaker 1976). We appear to be on some w_i, and we can easily imagine some other w_j, and we can ask: is w_i *actual* in a fundamental sense, in a way that w_j is not, in a way that could be picked out by an independent omniscient observer, in a way that it ought to be included in the fundamental ontology? Or is the thing which makes w_i "actual" and w_j merely "possible" the fact that this sentence was written in w_i? Bohmians should side with Stalnaker on this point and Everettians with Lewis.

So it seems to me that Bohm and Everett are two sides of the same coin, that their disagreements reduce to a bare question of metaphysics, expressible without any reference to the particulars of our physics. Both theories share in the assertion that ρ represents some ontic feature of reality. They share in the assertion that the fundamental dynamical law of nature operates on this ρ. They share in the assertion that the dynamics of *this* world (when we get around to admitting that this world exists) are derivative of the dynamics of ρ. The only disagreement is whether the thing described by ρ, the thing

[11] I have hidden my wilder speculations as to the nature of this "guidance" in Appendix C where no one will read them.

which looks and moves like an uncountable set of other worlds like this one, actually *is* that way or merely *looks and moves* that way. The question is, in bare metaphysical terms, whether those "other worlds" are real and concrete like this one; or imaginary, fictional, ersatz.[12]

To make what is perhaps an obvious point, both Bohm and Everett tell the exact same story when viewed from configuration space: a cloud evolving continuously and deterministically, rippling and flowing and self-interfering and splitting off like so much $3n$-dimensional mitosis. The sole dispute is, to put it colorfully, whether to distinguish the marvelous point of our lived experience with a golden glimmer of god and soul or with a "you are here" sticker. Is the ontology (ρ, w^*) or just ρ?

This is in dramatic contrast with a collapse theory, which insists that patches of the cloud *suddenly disappear*, that they discontinuously collapse into nothingness, as soon as they become dynamically separable from the patch we're on. This strikes me as extremely artificial, motivated purely by an instinctive aversion to the idea of worlds different from ours. It strikes me that way whether you use a naive collapse postulate or a more robust one in which worlds are *constantly* disappearing, the cloud is *constantly* contracting, spontaneously and at random, with all epicycles tuned appropriately to conform with data. Moreover I see absolutely no need for the introduction of such a postulate, when the continuous theories make exactly the same predictions. If you are really so concerned about the patches that have split off from ours, why not take the Bohmian approach and call them empty?

More to the point, the addition of a collapse postulate does nothing to resolve the central discomfiting feature of the quantum theory: the patch of ρ or ψ which you have

[12] I of course don't intend to settle this debate, but I can offer my personal view. With due respect to Prof. Stalnaker, who advised me against this conclusion, I think the disagreement is a non-substantive matter of terminology. I think the traditional notion of "real" does not have a unique extension to the case of many worlds, whether modal or quantum. We may place the boundary of the "real" at the outer edge of *this* world, or at the outer edge of some broader collection or amalgam of worlds; both choices are consistent with the term's conventional denotation. I believe this is ultimately a *normative* question of where we *ought* to draw this boundary, how we *ought* to speak, and I think that normative question is (like many normative questions, in my view) irreducibly underdetermined. But of course that means I also believe the central question of ontology, the question of *what is real*, is in a rigorous sense underdetermined, like the Godel sentence in ZFC, and thus *in practice* normative, conventional, terminological; so I wouldn't dare get caught believing any of this outside of a footnote.

not yet managed to collapse away still appears to be diffuse and self-interfering. Collapse has not bought you a definite table or chair; you are still looking at a fuzzy distribution over definite tables and chairs, even if you've managed to reduce the fuzziness somewhat. The "quantum state" you are trying to ontologize was simply never meant to refer to the *actual*, realized state of an individual physical system, and any attempt to shunt it into this role will necessarily involve unnatural philosophical contrivances.

So that is the view I would promote: locally Bohmian, globally Everettian. Wherever we eventually land on the question of actuality, these two theories are evidently on the same side of the fence against the profligate collapse theories and should recognize as much. I would hope that these camps could set aside their differences and argue collectively for a continuous dynamics, until Copenhagen and its offspring have been relegated to the laboratory, recognized as the "mathematical models which are adequate for description of our experiments" that they are and were always intended to be.

VII.
PROBABILITY AND LAWFULNESS

So: you want to predict the future. You've obtained through various means a set of knowledge about the present configuration of reality and you'd like to project it forward into knowledge about the future configuration of reality. Ideally, you'd like to drop an array of field lines on configuration space, so that any world-state will simply fall toward its inevitable future, tracing out the unique lawful trajectory from a given initial condition. Based on our Newtonian experience of reality and all the accumulated wisdom of the natural sciences up until about a century ago, this seems like a perfectly reasonable goal.

I'm very sorry to tell you that it's not possible. That is not the type of universe we were dealt. In our actual universe (if the quantum theory is essentially correct; if the Schrodinger equation is true) the notion of a "lawful trajectory" through configuration space is not a well-formed one. The property of "lawfulness" in our universe, of accordance with the fundamental dynamical law, is not attributable to any individual trajectory $w_i : T \to \text{Config}$ through configuration space.

The thing to which a boolean Lawful($\bullet$) predicate *can* apply, the thing which can be said to evolve in accordance with the laws of physics, is instead this object ψ or ρ or $\mathfrak{B}$. What are we to make of this? The situation is, as usual, quite clear when expressed in the language of the $\mathfrak{B}$ notation:

The laws of nature apply not to individual worlds but to sets of worlds.

Whether you see these worlds (again) as concrete or merely as a useful fiction, the point remains.

So what are you to do, if you really must predict the future? You can try to take some measurements and narrow it down. You can write down a long sequence of observations of a quantum system, to any finite level of precision you like. And then the full set of dynamically relevant worlds will be densely concentrated in one minuscule speck of the configuration space, something that looks almost (if you're squinting) like a single point.

But when you press play on your predictive apparatus, that tiny speck of ρ-fluid will not move like a point. You will still not get your deterministic trajectory. The Madelung flow is not laminar; it's turbulent, like smoke billowing from a pipe, or water flowing in a rocky stream. As the laws of nature tick forward, the "fluid" will shimmer and churn and self-interfere, will spread out and decohere. You'll find the lawful evolution of the universe dictates that some of the worlds end up in R and some others end up in $\bar{R}$, and you will have no way of knowing at the initial time to which subset the actual world belongs.

If we had been so fortunate as to find that the deterministic evolution of a world-set were laminar, like water flowing down a smooth decline, we might be having an altogether different conversation. It would still be disconcerting that lawfulness applies only at the level of ρ, but we might at least be able to hack together a quasi-classical predictive apparatus. A laminar flow is, by definition, only infinitesimally sensitive to infinitesimal changes in the initial conditions, so we could in theory collect enough information about the present configuration, narrow it down enough, that the future configuration would be determinate to within any error bars we like.

But we were not so fortunate; the Bohmian trajectories are, famously, sensitively dependent on infinitesimal changes in initial and boundary conditions.

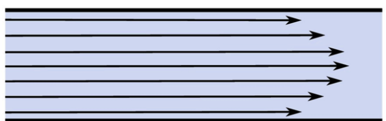

Figure 3: Laminar flow

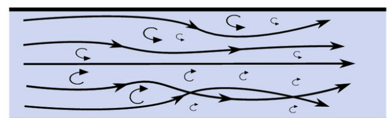

Figure 4: Turbulent flow

On the large scale, some facts about the future are certain: we are heading towards heat death, the Atlantic ridge is drifting apart, I am getting older. But on the small scale, the eddies predominate.

The fluid analogy continues to be clarifying. We imagine at time t_i a leaf is placed upstream of a fork in a river, and we know at some future time t_f it will be either in prong A or prong B. Imagine also that we have complete knowledge of the deterministic flow dynamics, such that we can at time t_i color each point in the pre-fork river either A (if that point will flow into prong A) or B (if that point will flow into prong B). For a laminar flow this coloring is neat and orderly, with a minimum "pixel size," so to speak, such that approximate knowledge of the initial leaf position is (except in edge cases) sufficient to predict which prong it will take. But for a turbulent flow, the coloring is fractal, such that no finite level of precision can deterministically project the future.

Of course, if there were an independent omniscient observer who could just tell you the exact position of the leaf, or the exact present configuration of our w_i, we could use our knowledge of the deterministic dynamical law to know the future. Unfortunately no such entity exists.

So you can keep making measurements, keep trying to narrow the space of possible worlds consistent with your observation down to a point, but that's just not how it works.

Any finite set of measurements will still be consistent with a region of configuration space with non-zero Lebesgue measure. You still won't know which trajectory you're on; you still can never know which of the possible worlds is the actual one. Here, I'll center and italicize it:

The future of the world which we observe is irreducibly underdetermined from any finite set of measurements of the present.

There is at the end of the day nothing more to say than this. I am genuinely sorry if you thought things would work out differently.

I have little patience for the oft-repeated criticism of many-worlds theories that they "have no room in them for *ignorance about the future*" (e.g. Albert 2015). Of course they do. They offer a very straightforward explanation for *our* ignorance of the future: we do not know which trajectory w_i is actual. If you take "actual" to be an indexical term, this is the self-locating uncertainty of e.g. Carroll, Sebens, and Vaidman; if you take "actual" to be a distinct feature of reality, this is the theory of Bohm. What they don't have room for is *God's* ignorance of the future, but I'm not sure why we should expect them to.

This, moreover, strikes me as the *only explanation* of what we call "probability" which makes even a modicum of sense, which doesn't merely kick the can down the road to some further unexplained principle of exogenous "chancy" influence.[13] A probabilistic claim is in the first place a modal claim, a claim that there are many ways the future could play out and that we don't yet know which will obtain. To speak probabilistically is inherently to invoke a plurality of worlds (a world where you win the lottery and a world where you don't; a world where you measure z-spin up and a world where you measure z-spin down) and to assign a numerical value to each. That is of course exactly what a many-worlds theory does. For any event space which divides Config at time t

[13] I remain entirely confused about the basic *what* and *how* of a fundamental "chance-law," and find it extremely fishy that this approach is considered the *de facto* null hypothesis, to be taught to physics students until another conception of probability proves its case beyond reasonable doubt.

into disjoint "possibilities" R_i, we assign to each event the numerical quantity

$$m(R_i, t) = \int_{R_i} \rho = \int_{R_i} |\psi|^2$$

These quantities are real numbers between 0 and 1 and they satisfy the Kolmogorov axioms. I'm not sure what else you were looking for.

The notion of probability as nothing more than a measure on a set of possible worlds is unsettling to some who would prefer to think of their behavior under conditions of absolute uncertainty as *rational*, which is to say, systematically derivable from the positive facts. If to say that an event has a 1% chance of occuring is merely to say that it occurs in 1% of a set of worlds of which this is one, then it is to say *nothing whatsoever* about what will actually transpire in this particular world. If probabilistic facts are simple matters of *proportion* or *mass*, if m-facts and ρ-facts are indeed facts about the structure of the universe-object, then they offer absolutely no guidance on what we ought to expect, how we ought to bet, when we ought to buy flood insurance or confirm or disconfirm a scientific theory. I think this point is exactly correct.

But I'm afraid I never expected the fundamental laws of physics to offer any normative guidance on our personal behavior. They are statements about what exists, not what we ought to do about it. If one happens to be an epistemic agent who frequently finds it necessary to make decisions under conditions of uncertainty, one is perfectly welcome to adopt a normative postulate for assigning credences—the Born rule, say,[14] or the principle of indifference. One is also perfectly welcome to not do any such thing, and to simply look toward the underdetermined future and say, *I have no clue.*

Ultimately the only conflict here is the conflict between what we wanted and what we got. There was an ambitious project to derive a formula by which the unique future trajectory of our lived experience could be mechanistically determined. That project concluded in failure when it was discovered that our lived experience is dynamically

[14] These things aren't magic; recall that $|\psi|^2$ is originally defined as the observed frequency, so this principle is essentially a special case of "guess/bet based on past frequencies."

inseparable from the turbulent behavior of a much larger structure existing outside the ambit of our epistemic capabilities. Great efforts have been made by the allies of this project to retrofit their discovery into the language of their prior expectations, to maintain the narrative that the advancement of science has brought humankind to a privileged and distinctive position in relation to the whole of existence, when in reality it seems that almost exactly the opposite is true.

APPENDIX A.
DETAILED DERIVATION OF MADELUNG'S EQUATIONS

We begin with the non-relativistic time-dependent Schrodinger equation in position basis:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\psi$$

For convenience we use units where $\hbar = m = 1$. We will also use subscript notation for partial derivatives, following the conventions of contemporary fluid mechanics.

$$i\partial_t\psi = -\tfrac{1}{2}\nabla^2\psi + V\psi$$

Rearrange and multiply by 2.

$$\nabla^2\psi - 2V\psi - 2i\,\partial_t\psi = 0$$

Substituting $\psi = \sqrt{\rho}\,e^{i\theta}$ we have

$$\nabla^2\left(\sqrt{\rho}\,e^{i\theta}\right) - 2V\sqrt{\rho}\,e^{i\theta} - 2i\,\partial_t\left(\sqrt{\rho}\,e^{i\theta}\right) = 0$$

The Laplacian expands into

$$\begin{aligned}
\nabla \cdot \nabla \left(\sqrt{\rho}\, e^{i\theta}\right) &= \nabla \left(e^{i\theta}\, \nabla \sqrt{\rho} + \sqrt{\rho}\, \nabla e^{i\theta}\right) \\
&= \nabla \left[e^{i\theta}\, (\nabla \sqrt{\rho} + i \sqrt{\rho}\, \nabla \theta)\right] \\
&= i e^{i\theta} \nabla \theta\, (\nabla \sqrt{\rho} + i \sqrt{\rho} \nabla \theta) + e^{i\theta} \left(\nabla^2 \sqrt{\rho} + i \sqrt{\rho}\, \nabla^2 \theta + i\, \nabla \sqrt{\rho}\, \nabla \theta\right) \\
&= e^{i\theta} \left[\nabla^2 \sqrt{\rho} - \sqrt{\rho} (\nabla \theta)^2 + i \sqrt{\rho}\, \nabla^2 \theta + 2i \nabla \sqrt{\rho}\, \nabla \theta\right]
\end{aligned}$$

and the partial derivative becomes

$$\partial_t \left(\sqrt{\rho} e^{i\theta}\right) = e^{i\theta} \partial_t \sqrt{\rho} + i e^{i\theta} \sqrt{\rho}\, \partial_t \theta$$

Thus the Schrodinger equation may be expanded into

$$\begin{aligned}
& e^{i\theta} \left[\nabla^2 \sqrt{\rho} - \sqrt{\rho} (\nabla \theta)^2 - 2V \sqrt{\rho} + 2 \sqrt{\rho}\, \partial_t \theta\,\right] \\
& \qquad + i e^{i\theta} \left[\sqrt{\rho}\, \nabla^2 \theta + 2 \nabla \sqrt{\rho}\, \nabla \theta - 2 \partial_t \sqrt{\rho}\right] = 0
\end{aligned}$$

Dividing through by $e^{i\theta}$ and separating the real and imaginary parts yields a pair of partial differential equations. We now make the final change of variable $u = -\nabla \theta$. The imaginary component becomes

$$\begin{aligned}
\sqrt{\rho}\, \nabla^2 \theta + 2 \nabla \sqrt{\rho}\, \nabla \theta - 2 \partial_t \sqrt{\rho} &= 0 \\
-\sqrt{\rho}\, \nabla u - 2u \nabla \sqrt{\rho} - 2 \partial_t \sqrt{\rho} &= 0 \\
(\rho \nabla u + 2u \sqrt{\rho}\, \nabla \sqrt{\rho}) + (2 \sqrt{\rho}\, \partial_t \sqrt{\rho}) &= 0 \\
\nabla \left(\sqrt{\rho}^{\,2} \cdot u\right) + \partial_t \left(\sqrt{\rho}^{\,2}\right) &= 0 \\
\nabla (\rho u) + \partial_t \rho &= 0 \qquad (*)
\end{aligned}$$

The real component becomes

$$\begin{aligned}\nabla^2\sqrt{\rho} - \sqrt{\rho}(\nabla\theta)^2 - 2V\sqrt{\rho} + 2\sqrt{\rho}\,\partial_t\theta = 0\\ -\frac{\nabla^2\sqrt{\rho}}{2\sqrt{\rho}} + \frac{(\nabla\theta)^2}{2} + V - \partial_t\theta = 0\end{aligned}$$

Using the Bohmian quantum potential $Q = -\frac{\nabla^2\sqrt{\rho}}{2\sqrt{\rho}}$ we have

$$\begin{aligned}Q + V &= \partial_t\theta - \tfrac{1}{2}(\nabla\theta)^2\\ -\nabla(Q+V) &= -\nabla\partial_t\theta + \tfrac{1}{2}\nabla\left[(\nabla\theta)^2\right]\\ -\nabla(Q+V) &= \partial_t u + \tfrac{1}{2}\nabla(u^2)\\ -\nabla(Q+V) &= \partial_t u + u\nabla u \qquad (**)\end{aligned}$$

The equation $(*)$ is a continuity equation, while $(**)$ resembles a balance of momentum equation. In classical fluid dynamics it is common to let

$$\frac{Dy}{Dt} = \partial_t y + u\nabla y$$

denote the material derivative of a variable y, which is to say the time rate of change of y at a point *following the flow described by* u rather than fixed in space. Thus we have the Madelung equations:

$$\boxed{\begin{aligned}\partial_t\rho &= -\rho\,\nabla u\\ \frac{Du}{Dt} &= -\nabla(Q+V)\end{aligned}}$$

Compare with the Euler conservation equations for a compressible fluid:

$$\begin{aligned}\partial_t\rho &= -\rho\,\nabla u\\ \frac{Du}{Dt} &= -\frac{\nabla p}{\rho} + g\end{aligned}$$

Thus taken literally as a description of a hydrodynamic system, the Schrodinger equation in Madelung form depicts an incompressible fluid acted on by "body accelerations" $g = -\nabla V$ and "internal mechanical pressure" $p = \rho Q$.

APPENDIX B.
DECOMPOSITION OF ρ INTO B

The following proofs have not been reviewed and may contain errors. I conjecture that the results extend to an aribtrary ρ with $\int \rho = 1$ but I only offer proofs for ρ which satisfy the Madelung equations. Most of the work in this case has already been done by Madelung and Bohm.

Proof of existence claim. Fix $t_0 \in T$. Let $B_0 = \mathbb{R}^n$ have measure $m_0 : \Sigma_{B_0} \to [0, 1]$ where Σ_{B_0} is the σ-algebra generated by open rectangles of $\mathbb{R}^n$, given by

$$m_0(R) = \int_R \rho(x, t_0)\,\mathrm{d}V$$

Thus by construction B_0 models ρ at time t_0. All that remains to do is project each point of B_0 forward in time by the Madelung–Bohm flow, maintaining the measure.

Define the local flow $\Phi(x, t) : \mathbb{R}^n \times T \to \mathbb{R}^n$ uniquely by

- $\Phi(x, t_0) = x$ for all $x \in \mathbb{R}^n$
- $\dfrac{\partial \Phi}{\partial t} = \vec{u}$ at every $(x, t) \in \mathbb{R}^n \times T$

so that for each x_0, the path $\Phi(x_0, t) : T \to \mathbb{R}^n$ is a Bohmian trajectory. Also let $\mathrm{img}(R, t)$ denote the image of a region R under the transformation $x \mapsto \Phi(x, t)$. By the continuity equation we must have

$$\int_R \rho(x, t_0)\,\mathrm{d}V = \int_{\mathrm{img}(R,t)} \rho(x, t)\,\mathrm{d}V$$

for all R, t.

We claim that $B = \{\Phi(x_0, t) : T \to \mathbb{R}^n\}_{x_0 \in \mathbb{R}^n}$ is the desired bundle, with measure carried over from m_0, that is, $m(X) = m_0(\{w(t_0) \mid w \in X\})$. To confirm the modeling relation, note

$$\begin{aligned} m(\mathrm{img}(R), t) &= m(\{w \in B \mid w(t) \in \mathrm{img}(R)\}) \\ &= m_0(\{w(t_0) \mid w(t) \in \mathrm{img}(R)\}) \\ &= m_0(R) \\ &= \int_R \rho(x, t_0)\, \mathrm{d}V \\ &= \int_{\mathrm{img}(R,t)} \rho(x, t)\, \mathrm{d}V \end{aligned}$$

as desired. As every Borel set is a countable union of open rectangles, and open sets are conserved under img^{-1}, the claim is proven for all R, t. □

Proof of uniqueness claim. ρ is uniquely recovered from B by

$$\rho(x, t) = \lim_{R \to x} \frac{m(R, t)}{\mathrm{vol}(R)}$$

where $\mathrm{vol}(R)$ is the standard Lebesgue measure. □

APPENDIX C.
SPECULATION AS TO THE NATURE OF THE INTERFERENCE TERM

If we are taking seriously the possibility that our particle-world is one “slice” of a larger physical entity, it is worthwhile to consider exactly how the motions of this entity are manifested in the dynamics of our particular world. Given that on this conception our world is essentially Bohmian, the clearest line of approach to this question is by investigating the Bohmian equation of motion for a particle:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\frac{1}{m}\nabla(Q + V)$$

This is of course identical to the classical Newtonian force law, except for the introduction of the new "quantum potential" term Q.

Simple enough, but there's a bit of funny business. The potential Q is a function of ρ, which is a feature of a *full configuration of reality.* Thus the classical potential $V(x)$ guides our particle toward a particular type of position, but the quantum potential $Q(c)$ guides it toward a particular type of *world-state.* In Bohm's phrasing, "the particles are guided in a correlated way" (1993).

It will be easier to understand the nature of this bizarre and apparently nonlocal "correlation" if we work with a simplified example. Imagine an alternate Bohmian-esque reality in which the quantum potential term is instead given by $\hat{Q}(c) = \rho(c)$.

In this reality, each particle still falls toward a classically low-energy position by the influence of the classical potential $V(x)$. But the *universe as a whole* avoids falling into a configuration with high $\rho(c)$. This avoidance is carried out simultaneously and in unison by every particle. A particle acts as if there is some mysterious *quantum force* pushing it away from any position where it would form (together with all other particles) a high-ρ configuration. There would seem to be, in effect, a physical resistance of the world to entering a state of high ρ-density, as if by an unseen pressure gradient.

In the context of configuration space, this looks to be a literal pressure gradient on worlds, essentially diffusive in character, whereby a world is physically repulsed by the presence of a high concentration of "adjacent" worlds (to borrow the spatial term). The upshot of this quantum correction term $\hat{Q}$ is to make the worlds of ρ "spread out" in configuration space, rather than all converging on the classical equilibrum.

What does that look like in an individual Bohmian world? We can imagine again the case of a single electron in a Coulomb potential, moving freely according to the Bohmian force law. The electron will not fall into the nucleus, despite that being the point of lowest classical potential V, because $\hat{Q} = \rho$ is too high there. The electron would instead appear to be physically repulsed by its "adjacent" counterparts, exactly as if by a physical pressure gradient; the electron together with its counterparts would form a spherical "electron cloud" gradient around the nucleus.

But, sadly, we were not dealt this simple quantum potential $\hat{Q} = \rho$. We were dealt:

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}$$

So the nature of the actual quantum force is not a simple pressure gradient pushing the world away from configurations of high ρ. The actual guidance condition, the actual quantum correction term, pushes the world away from configurations of high $-\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}$.

Which means... what, exactly? Let me read this mathematical expression in plain English. (That is more or less the point of the philosophy of physics, is it not?)

If the local curvature of $\sqrt{\rho}$ is positive, for instance if we are near a configuration which is a local minimum in $\sqrt{\rho}$, then we are being pushed toward configurations of low $\sqrt{\rho}$, the push increasing in strength as we approach the local minimum. But if the local curvature of $\sqrt{\rho}$ is *negative*, for instance if we are near a configuration which is a local *maximum* in $\sqrt{\rho}$, then we are being pushed toward configurations of *high* $\sqrt{\rho}$, but the push *decreases* in strength as we approach the local maximum.

It's an awful mouthful, but if you read it over a few times and try to picture what's going on, you see that the dynamics of such a system are periodic in character, wherein local minima rapidly become local maxima and local maxima rapidly become local minima. This of course explains the periodicity of simple quantum systems, and why it is so mathematically convenient to model the dynamics of such systems with an object of the form $\sqrt{\rho}\cos\theta + i\sqrt{\rho}\sin\theta$. In the real-world example of a single electron in a Coulomb potential (a hydrogen atom) I am told the eigenstates resemble a 3D analog of the resonances of a vibrating plate, though I have not confirmed these calculations myself.

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